Exercise 3.3.2

For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients:

(a) $f(x) = \cos \pi x/L$
[Verify formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

(d) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

Solution

Assume that $f(x)$ is a piecewise smooth function on the interval $0 \leq x \leq L$. The odd extension of $f(x)$ to the whole line with period $2L$ is given by the Fourier sine series expansion,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L},$$

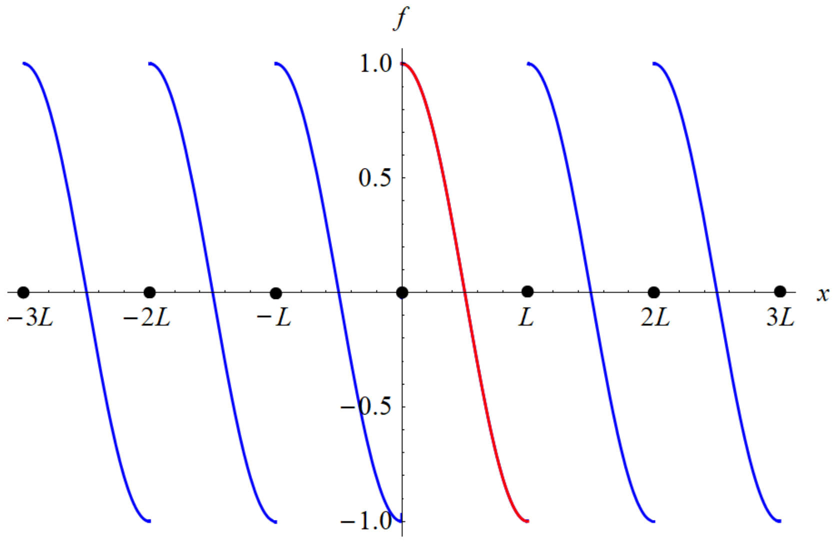
at points where $f(x)$ is continuous and by the average of the left-hand and right-hand limits at points of discontinuity. The coefficients B_n are obtained by multiplying both sides by $\sin \frac{p\pi x}{L}$ (p being an integer), integrating both sides with respect to x from 0 to L, and taking advantage of the fact that sine functions are orthogonal with one another.

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

Part (a)

For $f(x) = \cos \pi x/L$, the coefficients are

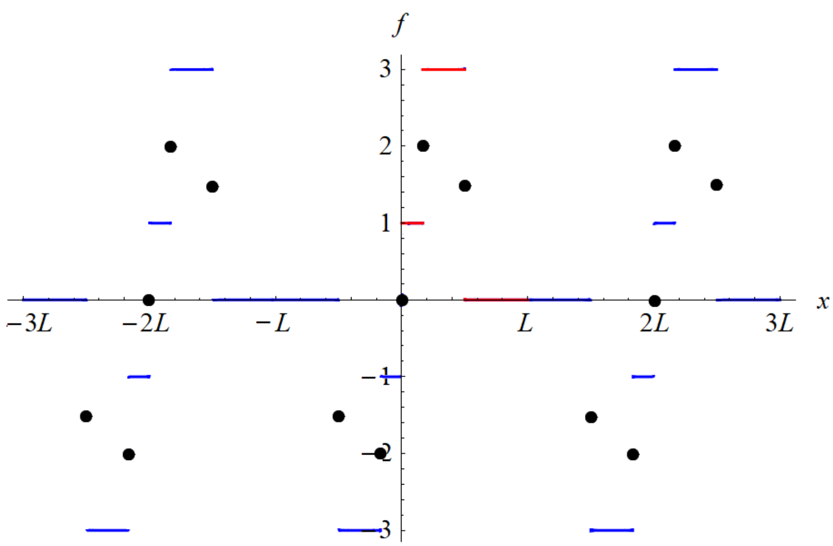
$$\begin{aligned}
B_n &= \frac{2}{L} \int_0^L \cos \frac{\pi x}{L} \sin \frac{n\pi x}{L}\, dx \\
&= \frac{2}{L} \int_0^L \frac{1}{2} \left[\sin\left(\frac{\pi x}{L} + \frac{n\pi x}{L}\right) - \sin\left(\frac{\pi x}{L} - \frac{n\pi x}{L}\right)\right] dx \\
&= \frac{1}{L} \left[\int_0^L \sin \frac{(1+n)\pi x}{L}\, dx - \int_0^L \sin \frac{(1-n)\pi x}{L}\, dx\right] = 0 \quad \text{if } n = 1 \\
&= \frac{1}{L} \left[\frac{[1+(-1)^n]L}{(1+n)\pi} - \frac{[1+(-1)^n]L}{(1-n)\pi}\right] \quad \text{if } n \neq 1 \\
&= \begin{cases} 0 & n = 1 \\ \dfrac{2[1+(-1)^n]n}{(n^2-1)\pi} & n \neq 1 \end{cases} \\
&= \begin{cases} 0 & n \text{ odd} \\ \dfrac{4n}{(n^2-1)\pi} & n \text{ even} \end{cases}.
\end{aligned}$$



Part (b)

For $f(x) = 1$ if $x < L/6$ and $f(x) = 3$ if $L/6 < x < L/2$ and $f(x) = 0$ if $x > L/2$, the coefficients are

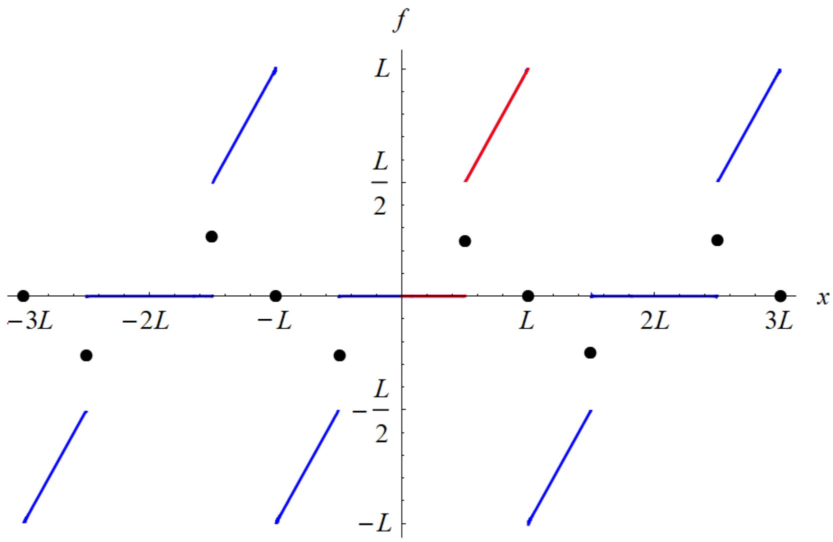
$$\begin{aligned} B_n &= \frac{2}{L}\left(\int_0^{L/6} \sin\frac{n\pi x}{L}\,dx + \int_{L/6}^{L/2} 3\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 0\sin\frac{n\pi x}{L}\,dx\right) \\ &= \frac{2}{n\pi}\left(1 + 2\cos\frac{n\pi}{6} - 3\cos\frac{n\pi}{2}\right). \end{aligned}$$



Part (c)

For $f(x) = 0$ if $x < L/2$ and $f(x) = x$ if $x > L/2$, the coefficients are

$$B_n = \frac{2}{L}\left(\int_0^{L/2} 0 \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} x \sin\frac{n\pi x}{L}\,dx\right) = \frac{L}{n^2\pi^2}\left\{n\pi\cos\frac{n\pi}{2} - 2\left[(-1)^n n\pi + \sin\frac{n\pi}{2}\right]\right\}.$$



Part (d)

For $f(x) = 1$ if $x < L/2$ and $f(x) = 0$ if $x > L/2$, the coefficients are

$$B_n = \frac{2}{L}\left(\int_0^{L/2} \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} 0 \sin\frac{n\pi x}{L}\,dx\right) = \frac{4}{n\pi}\sin^2\frac{n\pi}{4}.$$